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Excited Baryons in Large N_c QCD Revisited: The Resonance Picture Versus Single-Quark Excitations

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We analyze excited baryon properties via a $1/N_c$ expansion from two perspectives: as resonances in meson-nucleon scattering, and as single-quark excitations in the context of a simple quark model. For both types of analysis one can derive novel patterns of degeneracy that emerge as $N_c \rightarrow \infty$, and that are shown to be compatible with one another. This helps justify the single-quark excitation picture and may give some insight into its successes. We also find that in the large N_c limit one of the S_{11} baryons does not couple to the π - N channel but couples to the η - N channel. This is empirically observed in the $N(1535)$, which couples very weakly to the π - N channel and quite strongly to the η - N channel. The comparatively strong coupling of the $N(1650)$ to the π - N channel and weak coupling to η - N channel is also predicted. In the context of the simple quark model picture we reproduce expressions for mixing angles that are accurate up to $O(1/N_c)$ corrections and are in good agreement with mixing angles extracted phenomenologically.

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I. INTRODUCTION

The intellectual history of the quark model is a study in irony. In the 1960's the quark model provided a concrete dynamical model that incorporated $SU(3)$ flavor in a natural way, and thus organized the known hadrons according to an intelligible scheme. As such, it provided an essential role in the evolution of ideas that ultimately led to the formulation of QCD in the early 1970's as the underlying theory of strong-interaction dynamics. However, once QCD was discovered, the status of the quark model became quite problematic, since it was not known how this model was connected to QCD, as there was no known way to derive the quark model from QCD as some type of approximate treatment of the full theory. This irony would be of interest only to historians of science, if not for the fact that fully three decades since the QCD Lagrangian was first written down the quark model remains the principal tool used by the community for describing hadronic resonances. The reason is simple: Direct *ab initio* calculation of excited states from QCD remains an exceptionally difficult problem (although new lattice techniques are showing some promise [1]).

This paper focuses on the relationship of the quark model treatment of excited baryons to QCD, using techniques based on the $1/N_c$ expansion [2, 3]. We study excited baryons because the contracted $SU(2N_f)$ spin-flavor symmetry that emerges for baryons in large N_c [4, 5] provides a powerful tool for comparing results known to be generally true with those arising in the quark model. This analysis is useful for two reasons: i) It helps to justify the quark model, which is of utility since the quark model is the basis of considerable intuition about the excited states; and ii) it gives direct insight into certain aspects of the phenomenology of excited states.

The usefulness of the $1/N_c$ expansion for analyzing the properties of baryons in the ground-state band (*i.e.*, the N 's and Δ 's) has been demonstrated for numerous observables including masses, electromagnetic moments, and axial couplings [5, 6, 7, 8, 9, 10, 11, 12], with significant recent interest in properties of the Δ and the N - Δ transition [13, 14, 15, 16, 17]. The spin-flavor symmetry gives significant insight into the validity of the quark model. On one hand it is derived by imposing consistency between the N_c scaling of the π - N coupling ($N_c^{1/2}$) and that of π - N scattering (N_c^0) [4, 5]. On the other hand, the simplest way to compute the consequences of the symmetry is to treat the baryon as though it were a collection of N_c quark fields in identical S -wave single-particle orbitals and then keep track of the color, spin, and flavor [8, 10, 11], with the quarks combined into a color singlet and into a particular representation of combined spin-flavor symmetry. The N_c counting can then be implemented by introducing operators that break the symmetry and are accompanied by a characteristic power of $1/N_c$. This organizes operators contributing to a particular observable according to a well defined power-counting scheme.

The key point is that for these ground-state band baryons, large N_c QCD has the same spin-flavor symmetry as the simple quark model and has the same pattern of symmetry breaking. Thus, at least for those ground-state baryon properties that are insensitive to the dynamical details and are essentially fixed by the symmetries, large N_c QCD provides a justification of the quark model. In a similar way we wish to investigate the extent to which large N_c QCD treats excited baryons in an analogous manner. That is, we wish to understand what large N_c rules can be obtained from large N_c consistency rules at the purely hadronic level, and compare these results to what is obtained for a quark picture, to see what aspects of the quark picture can be justified.

A central issue in using the $1/N_c$ expansion to study quantities in our $N_c=3$ universe is whether to include quantities that vanish for $N_c=3$. For example, should the quantity $1-(3/N_c)$ be evaluated by immediately taking the $N_c \rightarrow \infty$ limit and retained as unity, or should one note that eventually one imposes $N_c=3$ in the $1/N_c$ expansion and treat it as vanishing? This question was discussed in the case of baryon charge radii and quadrupole moments in Ref. [15]. In the present context we believe it is useful to work in the $N_c \rightarrow \infty$ world directly, because we are attempting to make *qualitative* comparisons between two different pictures whose large N_c behaviors are known. We caution, however, that it is by no means yet entirely clear how to handle $1/N_c$ corrections to the two pictures in a consistent way. This issue is particularly thorny because there exist states at large N_c that do not exist at large N_c , and the role of these large N_c artifacts must be isolated before one attempts to draw phenomenological conclusions.

Before starting, we must note a key distinction between excited baryons and their ground-state band cousins. The ground-state baryons are stable in the large N_c limit (Of course, in the real world the Δ decays due to the anomalous lightness of the π , thanks to approximate chiral symmetry), while excited baryons are all resonances. Now, from standard counting based on Witten's original arguments [3], it is clear that the characteristic N_c scaling for the excitation energy of an excited baryon is N_c^0 , while the scaling of the three-point coupling between the excited baryon, a ground-state band baryon, and a meson is also N_c^0 . Thus, the resonance width also scales as N_c^0 . In this respect, baryons in large N_c QCD are fundamentally different from mesons; in the meson case we know that widths scale as $1/N_c$, so that well-defined narrow meson states exist at large N_c . Indeed, from the perspective of large N_c counting alone, one must be agnostic about the very existence of baryon resonances that are narrow enough to isolate. Here we simply note that the empirical evidence indicates identifiable resonances.

From the purely hadronic perspective the fact that excited baryons are associated with resonances simply suggests that the appropriate first step is to describe scattering processes, such as meson-nucleon scattering, in channels for which such resonances may reveal themselves. The role of large N_c QCD is then simply to relate scattering in various

channels [up to $O(1/N_c)$ corrections], in the sense that various linear combinations of channels are equal [18, 19, 20, 21]. We note that, while these relations were initially derived in the context of chiral soliton models, they are in fact model independent. An outline of a derivation of these relations directly from large N_c consistency relations along the lines of Ref. [8] appears in the Appendix. From these relations one can deduce patterns of degeneracy among resonances in various channels that are valid up to corrections of $O(1/N_c)$. We note that, although the linear relations in scattering amplitudes have been known for a long time, the patterns of mass degeneracies among the excited baryons reported here are brand new, their existence first mentioned in our recent Letter [22].

The fact that the excited baryons are resonances has always been an awkward fact for quark models. If one defines a quark model for a baryon as a description in which there are N_c constituent quarks interacting through potentials and with no mechanism for pair creation, then as a matter of principle there is no way that such a model can ever describe a physical state that is a meson-nucleon resonant scattering state. Implicitly, what is done is to assume that the quark model describes a state that is relatively weakly coupled via some quark pair-creation mechanism to a larger Hilbert space that includes the asymptotic two-hadron state. If such a coupling is weak enough, one expects the position of the resonance to be close to the bound-state energy of the uncoupled system. Thus, any quark model treatment that fits parameters so that the energies computed in the model match resonance masses implicitly makes a weak-coupling assumption. This is worth stressing in the present context, if only to remind ourselves that from a large N_c perspective there is no reason to assume such a coupling is weak (Note that this is not the case for mesons, where every type of pair creation is suppressed by $1/N_c$). Again, we assume here that the coupling is weak for reasons not connected to N_c , and proceed.

Let us look in a bit more detail at how the quark model for baryons is realized. The system can be solved as a true three-body problem, with the potential consisting of two-body or three-body interactions between the quarks that depend only on their relative coordinates. Such a treatment has the virtue of being consistent with the underlying spirit of the model and has the technical advantage that the center-of-mass coordinate automatically separates from the relative coordinates, allowing for a description of internal excitations that is not contaminated by any spurious center-of-mass motion. However, full three-body calculations are technically difficult. Moreover, the wave functions are complicated, and thus it is hard to obtain much intuition from them. Accordingly, many simple calculations are based on a single-particle potential-type model, where this potential is thought of as arising from the interactions of the other quarks. In the simplest version of this model (for $N_c=3$), the ground-state baryons have all three quarks in the lowest S -wave orbital (yielding a **56**-plet in spin-flavor for $N_f=3$) while the first excited group of baryons has one quark in a P wave and fills a mixed-symmetry **70**-plet spin-flavor state [23]. Both types of models can be called quark models, but for our purposes it is useful to distinguish between them. Accordingly, we refer to the second variant as the *quark-shell model*, since it has similarities with the shell model of nuclear physics.

A few additional comments about the quark-shell model are in order. First, much of the intuition many people have about excited baryons and much of the language used to describe them are based on the quark-shell model rather than more sophisticated treatments. The reason is that the simplicity of the model allows one to form a comprehensible picture of the state. Second, one may create more sophisticated versions of the quark-shell model that include admixtures of different single-particle descriptions, in order to include some of the correlations; such admixtures are called configuration mixing. Of course, if all possible configuration mixing is allowed and if the interactions being used are the full quark-quark potentials of the underlying quark model, then the quark-shell model is equivalent to the full three-body quark model, representing just a convenient basis in which to work rather than a distinct model (The situation is completely analogous to that of the case in the nuclear many-body problem [24]). Here, when we refer to the quark-shell model, we mean models in which configuration mixing is neglected or taken to be small. It is also worth noting that the simple quark-shell model (with little or no configuration mixing) does a good job of describing the spectrum of the lowest-lying observed N^* 's. In the present context, we note that to date all quark-based treatments that describe excited baryons in the large N_c limit of QCD were quark-shell models that neglect configuration mixing [25, 26, 27, 28, 29, 30, 31, 32]. In such a picture, the first excited states are either radial excitations of the symmetric ground-state multiplet (such as the Roper), or orbital excitations (with quantum number ℓ) of a single quark with respect to the other N_c-1 quarks remaining in a spin-flavor symmetrized “core.” Again using $SU(6)$ terminology, these states fill representations analogous to the three-color **56** and **70**, respectively [25, 26, 27, 28, 29, 30, 31, 32].

In this paper we compare the physical content of the two pictures—excited baryons as resonances in meson-nucleon scattering, and as single-quark excitations in a quark-shell model—to test whether the two are consistent with each other in a large N_c world. We consider the lowest positive- and negative-parity nonstrange excited baryon resonances. We find that generically the two pictures are compatible. That is, both pictures predict patterns of mass degeneracy at leading order in the $1/N_c$ expansion and these patterns are identical. We note that this compatibility is nontrivial and may help justify the use of the quark-shell model and explain its qualitative success.

We should comment briefly on the role of model dependence in what follows. The relations that follow from our treatment of meson-baryon scattering are truly model independent and are direct results of large N_c QCD. There is a

subtlety in results obtained for excited baryons using the quark model language: For the ground-state band, studies of the N_c dependence of operator matrix elements in a quark model picture [8] completely reproduce results of the large N_c consistency conditions. Something similar can be seen here—the multiplet structure we obtain using quark model language below is identical to that obtained by Pirjol and Yan [26] using large N_c consistency relations. In this sense, their results are model independent. However, both the analysis here and that of Ref. [26] are based on treatments of matrix elements of operators between excited baryon states. This is strictly only well defined for stable states. However, generically, as noted above the excited baryons are not stable in the large N_c limit; they have widths of $O(N_c^0)$.

Thus, the relations derived in quark model language are known to be valid and model independent only for stable excited states. One could imagine a world in which the quark masses and the pion were so heavy that the $N(1535)$ was stable. In such a world the quark model results and the large N_c consistency relations would agree for these stable states and would be truly model independent as $N_c \rightarrow \infty$. Unfortunately, with realistic quark masses there is no reason to think there are any stable baryon resonances at large N_c . This raises the question of whether any of results are in fact valid in the real world for the unstable baryons. Of course, it is not implausible that some results derived in a model-independent way assuming the states are stable may nevertheless be valid for the resonances. In essence, the question of whether that is true is at the heart of this paper. As we shall see, the multiplet structure of excited states seen in this quark model-type language, derived assuming that states are stable, are in fact seen in full large N_c QCD derived without this assumption. This is one of the principal results of this paper.

We also use the large N_c results to explore directly aspects of the phenomenology of the negative-parity baryons, and find two rather interesting phenomenological results. The first concerns the $N_{1/2}$ negative-parity states: The large N_c analysis predicts the existence of a $N_{1/2}$ negative-parity state whose coupling to the π - N channel is weak (vanishing at large N_c), but couples strongly to the η - N channel. In fact, $N(1535)$ has precisely this character. We similarly predict the other $N_{1/2}$ negative-parity state to couple weakly to the η - N channel while coupling strongly to the π - N channel, which is seen in the $N(1650)$. The second result concerns the mixing angles between the various excited nucleon states in the quark-shell model context. From large N_c emerges an analytic result predicting the value of these mixings, and we find that these predicted values are in good agreement with phenomenological extractions.

In Sec. II we discuss the resonance picture. The key point is the existence of model-independent linear relations between meson-nucleon scattering in various channels that become exact in the large N_c limit of QCD. These relations imply degeneracy patterns for excited baryons. In Sec. III we discuss the quark-shell model picture for the lowest-lying negative-parity baryons. This discussion is based on the methods of Refs. [28, 29]. However, we discover an important analytic result not elucidated in these works, namely, that at leading order in the $1/N_c$ expansion the mixing angles between various states are fixed and that various states are degenerate up to this order. Finally, in Sec. IV we discuss the implication of these results both in justifying the quark-shell model and directly in terms of phenomenology.

II. MESON-NUCLEON SCATTERING PICTURE

It has long been known, primarily through the work of Hayashi, Eckart, Holzwarth, and Walliser [18], and of Mattis and collaborators [19, 20, 21, 33], that the S matrices of various channels in meson-nucleon scattering (or more generally scattering of mesons off ground-state band baryons) are linearly related in the large N_c limit. For the present purpose it is sufficient to consider the case of π or η mesons scattering off a ground-state band baryon. In this case the S matrices are given by

$$S_{LL'RR'IJ}^\pi = \sum_K (-1)^{R'-R} \sqrt{(2R+1)(2R'+1)(2K+1)} \left\{ \begin{matrix} K & I & J \\ R' & L' & 1 \end{matrix} \right\} \left\{ \begin{matrix} K & I & J \\ R & L & 1 \end{matrix} \right\} s_{KL'L}^\pi, \quad (2.1)$$

$$S_{LRJ}^\eta = \sum_K \delta_{KL} \delta(LRJ) s_K^\eta. \quad (2.2)$$

For π scattering, the incoming baryon spin (which equals its isospin) is denoted as R , that of the final baryon is denoted R' , the incident (final) π is in a partial wave of orbital angular momentum L (L'), and I and J represent the (conserved) total isospin and angular momentum, respectively, of the initial and final states (and hence represent isospin and angular momentum of the state in the s channel). $S_{LL'RR'IJ}^\pi$ is the (isospin- and angular momentum-reduced) S matrix for this channel in the sense of the Wigner-Eckart theorem, the factors in braces are $6j$ coefficients, and $s_{KL'L}^\pi$ are universal amplitudes that are independent of I , J , R , and R' . For η -meson scattering, since $I_\eta = 0$ many of the quantum numbers are more tightly constrained. The isospin (= spin) R of the baryon is unchanged and moreover equals the total isospin I of the state. The orbital angular momentum L of the η remains unchanged in the process due to large N_c constraints, and J denotes the total angular momentum of the state, which is constrained by the triangle rule $\delta(LRJ)$. S_{LRJ}^η is the reduced scattering amplitude, and s_K^η are universal amplitudes independent

of J . The reason that various scattering amplitudes are linearly related is clear from the structure of Eqs. (2.1) and (2.2): There are more amplitudes $S_{LL'RR'IJ}^\pi$ than there are $s_{KL'L}^\pi$ amplitudes, and thus there are linear constraints between them that hold to leading order in the $1/N_c$ expansion; similarly, there are more S_{LRJ}^η amplitudes than s_K^η amplitudes.

Equations (2.1) and (2.2) are the starting point for our analysis of meson-nucleon resonances. These equations were first derived in the context of the chiral soliton model [18, 19, 20, 21, 33]. In this picture, the quantum number K has a simple interpretation—the soliton at the classical or mean-field level breaks both the rotational and isospin symmetries but preserves the length K of $\mathbf{K} \equiv \mathbf{I} + \mathbf{J}$. Thus, the Hamiltonian describing the intrinsic dynamics of the soliton (that not associated with collective zero modes) commutes with the “grand spin” \mathbf{K} , and excitations can be labeled by K . It is important to stress, however, that Eqs. (2.1) and (2.2) are exact results in large N_c QCD and are independent of any model assumptions. A derivation directly from the large N_c consistency rules [8] exploiting the famous $I_t = J_t$ rule [20] can be found in the Appendix.

The key point for our purposes is that, in order for a resonance to occur in one channel $S_{LL'RR'IJ}^\pi$ in π - N scattering, there must be a resonance in at least one of the contributing amplitudes $s_{KL'L}^\pi$. However, since the same $s_{KL'L}^\pi$ contributes to amplitudes in more than one channel, all of them resonate at the same energy, and this implies degeneracies in the excited baryon spectrum. An analogous argument holds for η - N scattering.

To make the preceding argument concrete, one needs a method to extract the resonance position from the S matrix for scattering. Here we adopt the usual theoretical prescription: One analytically continues the scattering amplitude in the complex plane to unphysical but on-shell kinematics and defines the complex resonance position to be the position of a pole in the scattering amplitude. One can then simply relate the real and imaginary parts to the mass and width of the resonance. Using this definition of the resonance position, the argument given above is quite clean. Analytically continuing Eqs. (2.1) and (2.2) to the complex plane, it is apparent that if there is a pole in the complex plane on the left-hand side for π - N scattering, then at this pole position the right-hand side must also diverge, implying a pole in one of the $s_{KL'L}^\pi$ amplitudes on the right-hand side. One can then turn this around and argue, as above, that since $s_{KL'L}^\pi$ contributes to multiple channels, all of them must resonate at the same point (unless other selection rules, *e.g.*, parity, forbid them to mix); similarly for η - N scattering. We note in passing that, although this theoretical definition of the resonance position is valid, there is a practical difficulty in extracting the precise resonance positions from scattering since one cannot directly probe unphysical kinematics, and thus there is always some model dependence in any extraction of the resonance position from data.

Consequences of Eqs. (2.1) and (2.2) for negative-parity partial waves can be seen in the right column of Table I, which lists the linear combinations of the $s_{KL'L}^\pi$ or s_K^η “ K -amplitudes” contributing to a particular partial wave of fixed I and J in the s channel. In using this table to deduce patterns of degeneracy of the baryon states, it is important to clarify whether any degeneracy might be expected in the $s_{KL'L}$ and s_K^η amplitudes themselves. On one hand it is clear that various K sectors ought to be dynamically distinct, since they are distinct in the large N_c limit. This is particularly clear in the context of the chiral soliton models, where the various K sectors are completely separate. More generally, there is no reason to suspect degeneracy between different K sectors, and it would be unnatural to impose any such degeneracies. On the other hand, it is quite plausible that there may be degeneracies in the poles of amplitudes with the same K but different values of L or L' : The orbital angular momentum of the π is not an immutable quantity in the same sense as I or J . Thus, if there is a resonance of fixed I and J but accessible by various L (*e.g.*, by scattering off a Δ rather than N), one would expect resonances in channels of different L to be degenerate. Similarly, scattering partial waves involving different mesons that nevertheless contain amplitudes in the same K channel (*e.g.*, s_{222}^π and s_2^η) can produce degenerate poles.

Examples of degenerate negative-parity multiplets at large N_c that one can infer from the meson-baryon scattering relations include:

$$N_{1/2}, \quad \Delta_{3/2}, \quad \cdots \quad (s_0^\eta), \quad (2.3)$$

$$N_{1/2}, \quad \Delta_{1/2}, \quad N_{3/2}, \quad \Delta_{3/2}, \quad \Delta_{5/2}, \quad \cdots \quad (s_{100}^\pi, s_{122}^\pi), \quad (2.4)$$

$$\Delta_{1/2}, \quad N_{3/2}, \quad \Delta_{3/2}, \quad N_{5/2}, \quad \Delta_{5/2}, \quad \Delta_{7/2}, \quad \cdots \quad (s_{222}^\pi, s_2^\eta), \quad (2.5)$$

$$\Delta_{3/2}, \quad N_{5/2}, \quad \Delta_{5/2}, \quad \Delta_{7/2}, \quad \cdots \quad (s_{322}^\pi), \quad (2.6)$$

where the states are listed on the left and the contributing amplitudes on the right. The ellipses indicate that in the large N_c world the multiplets are infinite dimensional, and we have simply listed the low-spin and -isospin members of the multiplet. The preceding multiplets are one of the principle results of this work. A few comments about them are in order. First note that at large N_c the poles of the scattering for these various members of the multiplet occur at the same point in the complex plane. Thus, the states have both the same mass and the same width as $N_c \rightarrow \infty$. Next, let us look at the multiplet in Eq. (2.4). Note that we have included the $\Delta_{5/2}$ for the s_{100} channel; this may be surprising in light of the fact that Table I has no partial wave for $\Delta_{5/2}$ with a contribution from s_{100} . It is because the table is restricted to partial waves in scattering off the N and Δ , while the s_{100} contribution to $\Delta_{5/2}$ is seen in

TABLE I: Negative-parity mass eigenvalues in the quark-shell model picture, corresponding partial waves, and their expansions in terms of K -amplitudes. The association of masses with states is from the large N_c quark-shell model relations in Sec. III. The superscripts πNN , $\pi N\Delta$, $\pi\Delta\Delta$, ηNN , and $\eta\Delta\Delta$ refer to the scattered meson and the initial and final baryons, respectively. The partial-wave amplitudes are derived from Eqs. (2.1) and (2.2). Note that these states are those appropriate to a large N_c world (As discussed in the text, some do not occur for $N_c=3$). We only list states with quantum numbers consistent with a single quark excited to $\ell=1$ and with total isospin of $3/2$ or less; and we only list partial waves sufficient to accommodate all the given resonances (hence in particular $L=L' \leq 2$).

State	Quark Model Mass	Partial Wave, K -Amplitudes
$N_{1/2}$	m_0, m_1	$S_{11}^{\pi NN} = s_{100}^{\pi}$ $D_{11}^{\pi\Delta\Delta} = s_{122}^{\pi}$ $S_{11}^{\eta NN} = s_0^{\eta}$
$\Delta_{1/2}$	m_1, m_2	$S_{31}^{\pi NN} = s_{100}^{\pi}$ $D_{31}^{\pi\Delta\Delta} = \frac{1}{10}(s_{122}^{\pi} + 9s_{222}^{\pi})$ $D_{31}^{\eta\Delta\Delta} = s_2^{\eta}$
$N_{3/2}$	m_1, m_2	$D_{13}^{\pi NN} = \frac{1}{2}(s_{122}^{\pi} + s_{222}^{\pi})$ $D_{13}^{\pi N\Delta} = \frac{1}{2}(s_{122}^{\pi} - s_{222}^{\pi})$ $S_{13}^{\pi\Delta\Delta} = s_{100}^{\pi}$ $D_{13}^{\pi\Delta\Delta} = \frac{1}{2}(s_{122}^{\pi} + s_{222}^{\pi})$ $D_{13}^{\eta NN} = s_2^{\eta}$
$\Delta_{3/2}$	m_0, m_1, m_2	$D_{33}^{\pi NN} = \frac{1}{20}(s_{122}^{\pi} + 5s_{222}^{\pi} + 14s_{322}^{\pi})$ $D_{33}^{\pi N\Delta} = \frac{1}{5\sqrt{10}}(2s_{122}^{\pi} + 5s_{222}^{\pi} - 7s_{322}^{\pi})$ $S_{33}^{\pi\Delta\Delta} = s_{100}^{\pi}$ $D_{33}^{\pi\Delta\Delta} = \frac{1}{25}(8s_{122}^{\pi} + 10s_{222}^{\pi} + 7s_{322}^{\pi})$ $S_{33}^{\eta\Delta\Delta} = s_0^{\eta}$ $D_{33}^{\eta\Delta\Delta} = s_2^{\eta}$
$N_{5/2}$	m_2	$D_{15}^{\pi NN} = \frac{1}{9}(2s_{222}^{\pi} + 7s_{322}^{\pi})$ $D_{15}^{\pi N\Delta} = \frac{\sqrt{14}}{9}(s_{222}^{\pi} - s_{322}^{\pi})$ $D_{15}^{\pi\Delta\Delta} = \frac{1}{9}(7s_{222}^{\pi} + 2s_{322}^{\pi})$ $D_{15}^{\eta NN} = s_2^{\eta}$
$\Delta_{5/2}$	m_0, m_1	$D_{35}^{\pi NN} = \frac{1}{90}(27s_{122}^{\pi} + 35s_{222}^{\pi} + 28s_{322}^{\pi})$ $D_{35}^{\pi N\Delta} = \frac{1}{90}\sqrt{\frac{7}{5}}(27s_{122}^{\pi} + 5s_{222}^{\pi} - 32s_{322}^{\pi})$ $D_{35}^{\pi\Delta\Delta} = \frac{1}{450}(189s_{122}^{\pi} + 5s_{222}^{\pi} + 256s_{322}^{\pi})$ $D_{35}^{\eta\Delta\Delta} = s_2^{\eta}$
$\Delta_{7/2}$	m_2	$D_{37}^{\pi\Delta\Delta} = \frac{1}{5}(2s_{222}^{\pi} + 3s_{322}^{\pi})$

scattering off a ground-state band baryon with $I = J = 5/2$, which though absent in our world, exists at large N_c . Note also that the degeneracy pattern seen in Eq. (2.4) occurs for both the s_{100} contributions and the s_{122} . There are two logical possibilities for this to occur: Either there are two distinct multiplets with the same degeneracy patterns, or the two multiplets are in fact the same. The second possibility is clearly more economical, and we believe it to be correct. Similarly, the multiplet pattern seen in Eq. (2.5) occurs for both π and η scattering, strongly suggesting that the same physical states occur in both. These observations tie in neatly with our previous discussion.

We should note that the degeneracy patterns shown above are newly derived in the context of meson-baryon scattering. This is somewhat surprising, since nearly 20 years ago Mattis and Karliner [33] computed the excited baryon spectrum in the Skyrme model directly from pion-baryon scattering using Eq. (2.1). In that work the only degeneracy found was between the $N_{1/2}$ and the $\Delta_{1/2}$. How can we understand these degeneracy rules in light of the fact that explicit calculations based on the same fundamental formula missed them? The answer lies in the algorithm used in Ref. [33] to extract resonance positions. The technique first computed the scattering amplitudes and then used motion in the Argand plots to fix the resonance position. This is essentially the technique used by experimentalists, and is highly appropriate when used in comparison with experimental extractions. However, by restricting attention to physical kinematics, one cannot directly access the pole position.

III. NEW RESULTS IN THE QUARK-SHELL MODEL PICTURE

Although the multiplet structure in Eqs. (2.3)–(2.6) is newly derived here in the context of meson-baryon scattering, the existence of such towers has been known for some time in the context of quark model-type treatments in the work of Pirjol and Yan [26]. Although the formalism used in Ref. [26] is apparently model independent, as discussed in the Introduction it makes use of a strong dynamical assumption characteristic of the quark model, namely that the excited states are stable. This assumption is generally not compatible with large N_c QCD. The derivation of these degenerate multiplets in [26] is quite beautiful conceptually, essentially applying the reasoning of Refs. [4, 5]. However, it is computationally somewhat involved. Here we rederive these results using a more explicit quark-shell model language (but an analogous dynamical assumption of stable baryons) using the methods of [28, 29].

As noted in the Introduction, our approach here is to work directly in the $N_c \rightarrow \infty$ world. This observation has a direct bearing on the enumeration of states in the quark-shell model. States in the first negative-parity multiplet ($\ell = 1$) have an $(N_c - 1)$ -quark core that is completely symmetric under spin \times flavor, and thus have the quantum numbers $S_c = I_c$ in the nonstrange case. Multiple states with the same fixed values of I and J are distinguished by the total spin S carried by the quarks, and this is denoted [28, 29] by primes (no primes for $S = 1/2$, one for $S = 3/2$, *etc.*). It is an elementary exercise in combining angular momenta and isospin to show that, for $N_c \geq 5$, the states with $I \leq 3/2$ are $N_{1/2}$, $N'_{1/2}$, $N_{3/2}$, $N'_{3/2}$, $N'_{5/2}$, $\Delta_{1/2}$, $\Delta'_{1/2}$, $\Delta_{3/2}$, $\Delta'_{3/2}$, $\Delta''_{3/2}$, $\Delta'_{5/2}$, $\Delta''_{5/2}$, and $\Delta'_{7/2}$. For $N_c = 3$, the states $\Delta'_{1/2}$, $\Delta'_{3/2}$, $\Delta''_{3/2}$, $\Delta'_{5/2}$, $\Delta''_{5/2}$, and $\Delta'_{7/2}$ do not occur, but must be included in a full $1/N_c$ analysis until the final step of setting $N_c = 3$.

Working up to $O(N_c^0)$ in the quark-shell model picture, one finds 3 operators contributing to the Hamiltonian, denoted in Refs. [28, 29] as $\mathcal{H} = c_1 \mathbb{1} + c_2 \ell s + c_3 \ell^{(2)} g G_c / N_c$. To remind the reader (Refer to Refs. [28, 29] if the following notation is not familiar), lowercase indicates operators acting upon the excited quark, and subscript c indicates those acting upon the core. G^{ia} denotes the combined spin-flavor operator $\propto q^\dagger \sigma^i \tau^a q$, and $\ell^{(2)}$ is the $\Delta\ell = 2$ tensor operator. This is to be contrasted with Refs. [26], which effectively include only one spin \times flavor-breaking operator at this order.

The Hamiltonian up to $O(N_c^0)$ for the mixed $N_{1/2}$ states reads

$$H_{N_{1/2}} = \left(\bar{N}_{1/2} \bar{N}'_{1/2} \right) \mathbf{M}_{N_{1/2}} \begin{pmatrix} N_{1/2} \\ N'_{1/2} \end{pmatrix}, \quad (3.1)$$

as may be obtained from Eqs. (A6)–(A8) or Table II of Ref. [29], again including only contributions up to $O(N_c^0)$. The mass matrix $\mathbf{M}_{N_{1/2}}$ is diagonalized by the unitary matrix $U_{N_{1/2}}$: $U_{N_{1/2}} \mathbf{M}_{N_{1/2}} U_{N_{1/2}}^\dagger = \text{diag} \left(M_{N_{1/2}}^{(1)}, M_{N_{1/2}}^{(2)} \right)$, where

$$\mathbf{M}_{N_{1/2}} = \begin{pmatrix} c_1 N_c - \frac{2}{3} c_2 & -\frac{1}{3\sqrt{2}} c_2 - \frac{5}{24\sqrt{2}} c_3 \\ -\frac{1}{3\sqrt{2}} c_2 - \frac{5}{24\sqrt{2}} c_3 & c_1 N_c - \frac{5}{6} c_2 - \frac{5}{48} c_3 \end{pmatrix}, \quad (3.2)$$

$$U_{N_{1/2}} = \begin{pmatrix} \cos \theta_{N_1} & \sin \theta_{N_1} \\ -\sin \theta_{N_1} & \cos \theta_{N_1} \end{pmatrix}; \quad (3.3)$$

$N_{1/2}$ and $N'_{1/2}$ refer to unmixed negative-parity spin-1/2 nucleon states in the initial quark-shell model basis. Anticipating a remarkable result, let us define 3 particular combinations of the parameters:

$$\begin{aligned} m_0 &\equiv c_1 N_c - \left(c_2 + \frac{5}{24} c_3 \right), \\ m_1 &\equiv c_1 N_c - \frac{1}{2} \left(c_2 - \frac{5}{24} c_3 \right), \\ m_2 &\equiv c_1 N_c + \frac{1}{2} \left(c_2 - \frac{1}{24} c_3 \right). \end{aligned} \quad (3.4)$$

One finds that $M_{N_{1/2}}^{(1)} = m_0$, $M_{N_{1/2}}^{(2)} = m_1$, and $\tan \theta_{N_1} = \sqrt{2}$. Note first that, had the numerical coefficients in Eq. (3.2) been arbitrary, the eigenvalues would in general contain square roots of terms quadratic in c_2 and c_3 , and the mixing angle would have been a complicated function of their ratio. The simplicity of the actual results indicates something deep is happening. Indeed, this simple mixing angle result at large N_c was earlier noticed in the work of Pirjol and Yan [26]. As will be seen shortly, results for states with other quantum numbers are equally simple.

Note that the simplicity of the present result—analytic expressions for both the masses and the mixing angle—was not noted in Refs. [28, 29]; the reason is simply that previous work always included the $O(1/N_c)$ terms in the Hamiltonian and then diagonalized numerically. The simple result given above, however, only holds to $O(N_c^0)$, and the inclusion of the $1/N_c$ correction terms in the Hamiltonian obscured the simple leading result.

Using an analogous notation for the other states,

$$\mathbf{M}_{N_{3/2}} = \begin{pmatrix} c_1 N_c + \frac{1}{3}c_2 & -\frac{\sqrt{5}}{6}c_2 + \frac{\sqrt{5}}{48}c_3 \\ -\frac{\sqrt{5}}{6}c_2 + \frac{\sqrt{5}}{48}c_3 & c_1 N_c - \frac{1}{3}c_2 + \frac{1}{12}c_3 \end{pmatrix}, \quad (3.5)$$

we find $M_{N_{3/2}}^{(1)} = m_2$, $M_{N_{3/2}}^{(2)} = m_1$ and $\tan \theta_{N_3} = -1/\sqrt{5}$. The spin-5/2 state is unmixed but also has a degenerate eigenvalue: $M_{N_{5/2}} = m_2$.

Next let us consider the Δ states:

$$\mathbf{M}_{\Delta_{1/2}} = \begin{pmatrix} c_1 N_c + \frac{1}{3}c_2 & +\frac{\sqrt{5}}{6}c_2 - \frac{\sqrt{5}}{48}c_3 \\ +\frac{\sqrt{5}}{6}c_2 - \frac{\sqrt{5}}{48}c_3 & c_1 N_c - \frac{1}{3}c_2 + \frac{1}{12}c_3 \end{pmatrix}, \quad (3.6)$$

which gives $M_{\Delta_{1/2}}^{(1)} = m_2$, $M_{\Delta_{3/2}}^{(2)} = m_1$, and $\tan \theta_{\Delta_1} = -1/\sqrt{5}$. Next,

$$\mathbf{M}_{\Delta_{3/2}} = \begin{pmatrix} c_1 N_c - \frac{1}{6}c_2 & +\frac{5}{6\sqrt{2}}c_2 + \frac{1}{48\sqrt{2}}c_3 & +\frac{\sqrt{3}}{16}c_3 \\ +\frac{5}{6\sqrt{2}}c_2 + \frac{1}{48\sqrt{2}}c_3 & c_1 N_c - \frac{2}{15}c_2 - \frac{1}{15}c_3 & -\frac{3}{10}\sqrt{\frac{3}{2}}c_2 - \frac{7}{80}\sqrt{\frac{3}{2}}c_3 \\ +\frac{\sqrt{3}}{16}c_3 & -\frac{3}{10}\sqrt{\frac{3}{2}}c_2 - \frac{7}{80}\sqrt{\frac{3}{2}}c_3 & c_1 N_c - \frac{7}{10}c_2 - \frac{7}{120}c_3 \end{pmatrix}, \quad (3.7)$$

with $M_{\Delta_{3/2}}^{(1)} = m_0$, $M_{\Delta_{3/2}}^{(2)} = m_2$, and $M_{\Delta_{3/2}}^{(3)} = m_1$. The mixing angles are parameterized by

$$U_{\Delta_{3/2}} = \begin{pmatrix} \cos \theta \cos \phi & \sin \theta \cos \phi & \sin \phi \\ -\cos \theta \sin \phi \sin \psi - \sin \theta \cos \psi & -\sin \theta \sin \phi \sin \psi + \cos \theta \cos \psi & \cos \phi \sin \psi \\ -\cos \theta \sin \phi \cos \psi + \sin \theta \sin \psi & -\sin \theta \sin \phi \cos \psi - \cos \theta \sin \psi & \cos \phi \cos \psi \end{pmatrix}, \quad (3.8)$$

and turn out to be $\tan \theta = -\sqrt{2}$, $\tan \phi = -1$, and $\tan \psi = -1/3$. $\Delta_{5/2}$ has

$$\mathbf{M}_{\Delta_{5/2}} = \begin{pmatrix} c_1 N_c + \frac{1}{5}c_2 + \frac{1}{60}c_3 & -\frac{\sqrt{21}}{10}c_2 + \frac{\sqrt{21}}{80}c_3 \\ -\frac{\sqrt{21}}{10}c_2 + \frac{\sqrt{21}}{80}c_3 & c_1 N_c - \frac{1}{5}c_2 + \frac{1}{15}c_3 \end{pmatrix}, \quad (3.9)$$

from which $M_{\Delta_{5/2}}^{(1)} = m_0$, $M_{\Delta_{5/2}}^{(2)} = m_1$, and $\tan \theta_{\Delta_5} = \sqrt{3/7}$. Finally, $\Delta_{7/2}$ is unmixed and has eigenvalue $M_{7/2} = m_2$. The pattern, masses being equal to one of three eigenvalues and mixing angles having simple expressions, obviously extends into the Δ sector. The compilation of mass eigenvalues into states of various quantum numbers is given in Table I.

Clearly, the fact that all of the masses described by the model are given by either m_0 , m_1 , or m_2 to leading order in the $1/N_c$ expansion implies that at large N_c the various states fall into degenerate multiplets. These multiplets are given by

$$N_{1/2}, \quad \Delta_{3/2}, \quad \cdots \quad (m_0), \quad (3.10)$$

$$N_{1/2}, \quad \Delta_{1/2}, \quad N_{3/2}, \quad \Delta_{3/2}, \quad \Delta_{5/2}, \quad \cdots \quad (m_1), \quad (3.11)$$

$$\Delta_{1/2}, \quad N_{3/2}, \quad \Delta_{3/2}, \quad N_{5/2}, \quad \Delta_{5/2}, \quad \Delta_{7/2}, \quad \cdots \quad (m_2), \quad (3.12)$$

where the states are listed on the left and the masses on the right.

After a draft of the present paper was completed we became aware of similar work by Pirjol and Schat [34], who obtained similar results and computed $1/N_c$ corrections.

IV. DISCUSSION

A. On the Compatibility of the Resonance and Quark-Based Pictures

The results of the previous two sections are quite striking. In both the resonance pole picture and the quark-shell model picture, one finds that the excited baryons are organized into multiplets of states that are degenerate modulo

splittings arising at next-to-leading order, $O(1/N_c)$. Let us compare the multiplet structures predicted by the two pictures as given in Eqs. (2.3)-(2.6) and Eqs. (3.10)-(3.12). It is clear that the multiplet structure in Eq. (2.3) is identical to that of Eq. (3.10); that in Eq. (2.4) is identical to that of Eq. (3.11); and that in Eq. (2.5) is identical to that of Eq. (3.12). The two pictures are compatible. The interpretation is quite simple: the multiplet states with mass m_0 in the quark-shell model are those states for which the resonance occurs in the $K = 0$ scattering channel and analogously for m_1 states ($K = 1$) and m_2 states ($K = 2$). This result is highly significant in justifying the quark model.

Before turning to the question of just how strong this justification is, we note that the resonance picture can have poles for $K = 3$, as seen in Eq. (2.6), for which we have not reported any analogous states in the quark-shell model. This in no way spoils the compatibility of the two pictures. Rather, it merely reflects the fact that the quark-shell model studies were limited to single-quark excitations in the lowest $\ell = 1$ orbital. Had we considered higher excitations in a quark-shell model, such as two-quark excitations or excitations in the $\ell = 3$ orbital, presumably we would have seen a degenerate multiplet consistent with Eq. (2.6). This prediction, that such a multiplet structure will be found higher in the spectrum of a large N_c quark-shell model, is a stringent test of our interpretation. Studies of higher excited states in the quark-shell model are presently under way [35].

While studies of higher negative-parity states remain to be completed, we have also computed the degeneracy patterns for the lowest-lying excited positive-parity states. These states include $N(1440)$ (P_{11}) and $\Delta(1600)$ (P_{33}). One finds, using Eq. (2.1), the relations

$$\begin{aligned} P_{11}^{\pi NN} &= (s_{011}^{\pi} + 2s_{111}^{\pi})/3, \\ P_{13}^{\pi NN} &= (s_{111}^{\pi} + 5s_{211}^{\pi})/6, \\ P_{31}^{\pi NN} &= (s_{111}^{\pi} + 5s_{211}^{\pi})/6, \\ P_{33}^{\pi NN} &= (2s_{011}^{\pi} + 5s_{111}^{\pi} + 5s_{211}^{\pi})/12. \end{aligned} \tag{4.1}$$

Since no poles are observed in the lowest multiplet for P_{13} or P_{31} , consistency between the resonance and quark-shell model pictures is achieved simply by placing a pole in s_{011}^{π} but not in s_{111}^{π} or s_{211}^{π} for this multiplet. We note that this result holds more generally, being valid for any state that in the quark-shell model lies in a spin-flavor symmetric multiplet.

The present interpretation—that quark shell model states correspond to a well-defined K quantum number (up to $1/N_c$ corrections)—is based on meson-baryon scattering, with scattering restricted to π -baryon and η -baryon channels. Of course, if the interpretation is correct, one should also find consistency between the multiplet structure of the quark-shell model and resonances deduced from mixed scattering, in which the initial meson is a π and final meson is an η . Although not presented here, it is straightforward to verify that this is true.

Let us now turn to the question of just how strongly the present result justifies the quark-shell model picture. We start by observing that, whatever justification there is for the quark-shell model at $N_c = 3$ should become increasingly reliable as N_c becomes large. The basic point is simply that there are more quarks available that combine to generate the effective single-body potential seen by the last quark. Thus, if there is justification for the model at $N_c = 3$, it is likely stronger for large N_c . Conversely, a clear failure of the model at large N_c would suggest that the picture is unlikely to be valid for $N_c = 3$. Thus, the fact that the quark-shell picture produces the same qualitative spectrum in terms of the multiplet structure as is seen in large N_c QCD from scattering is a real test, in the sense that the failure to do so would have cast serious doubts on the model.

Of course, the fact that the multiplet structure in the quark-shell model agrees with that of large N_c QCD does not justify all aspects of the model. In particular, it does not justify the dynamical details of the model in general, and certainly does not justify an approach that neglects open channels for decay. Rather, as for the ground-state band, it justifies those aspects of the model that essentially follow from the contracted $SU(2N_f)$ symmetry.

Finally, we note that compatibility of the excitation spectrum of the quark-shell model with large N_c QCD is highly nontrivial. At first blush, one might think the result *is* trivial. After all, Witten's initial derivation [3] of large N_c rules for baryons was done using heavy quarks, which essentially defines a quark model in the first sense described in the Introduction. Moreover, the counting used by Witten was essentially combinatoric and applied independently of the dynamical details. Thus, one expects that any relations that apply for large N_c QCD should hold in generic large N_c quark models.

However, as noted in the Introduction, we are *not* studying a full quark model; rather we are using a quark-shell model. Of course, one can again appeal to Witten's original argument and argue that the Hartree approximation emerging at large N_c is a single-particle picture in exactly the same manner as the quark-shell model. However, the preceding argument is specious. It is certainly true that the Hartree approximation becomes valid at large N_c *for the ground state*. Whether it is valid for excited states is a bit more subtle. It is well known [36] in many-body theory that mean-field theories such as the Hartree approach respect the underlying symmetries of the theory or spontaneously break them. However, the use of a mean-field potential for excited states, for example as in the Tamm-Dancoff

approximation, involves *ad hoc* truncations that generically violate the symmetries. In contrast, symmetry-conserving approximations to treat excited states, such as linear response theory or random phase approximation, typically go beyond the mean-field potential and take into account ground-state correlations. Since the results of our studies are essentially group theoretic—and thus entirely dependent on the treatment of the symmetries—it is by no means *a priori* obvious that the use of the mean-field potential for treatments of the excited states is adequate. Thus, the success of the quark-shell model in replicating the multiplet structure is quite significant. Again we note that it is important that the quark-shell model is justified (and not just the full treatment of the N_c -body quark model), as much intuition has been gleaned from the quark-shell model over the years.

B. Phenomenological Consequences

The results of Secs. II and III may be used to gain phenomenological insight into the low-lying excited baryons. A certain amount of care must be exercised when doing this, however, since the physical world of $N_c = 3$ cannot be regarded as an approximately large N_c world for all purposes. In particular, consider the most striking formal results of this work—the existence of nearly degenerate multiplets of states associated with a fixed K . Unfortunately, it will be very difficult to extract this structure directly in the baryon spectrum for the low-lying odd-parity states in the real world. The key difficulty concerns the scales of the problem: Note that the physical states in question vary in mass from 1520 MeV ($N_{3/2}$) to 1700 ($\Delta_{3/2}$), as listed by the Particle Data Group [37]. Thus, all of the observed states lie within a 200 MeV window. However, Δ - N mass splitting, a $1/N_c$ effect, is ~ 290 MeV. Thus, the actual splittings are not large enough to resolve, given the characteristic size of the $1/N_c$ effects. Studies of constraints on these next-to-leading $1/N_c$ effects are underway [35].

Fortunately, there are phenomenological predictions of the preceding analysis that may well be meaningful in the physical world. Consider Eqs. (2.3) and (3.10). With the interpretation above, we would say that the quark-shell states with mass m_0 in the large N_c world correspond to states accessible in scattering experiments characterized by modes with $K = 0$. Now, it so happens that π - N scattering does not couple to negative-parity $K = 0$ modes. This can be seen directly from the structure of the $6j$ coefficients in Eq. (2.1), which implies that $K \geq |L - 1|$. Clearly, $K = 0$ can only happen for $L = 1$, but $L = 1$ makes even-parity states. Thus, in Eq. (2.3) we see that the $K = 0$ multiplet is accessible via η - N scattering but not via π - N . Of course, this result that the $K = 0$ negative-parity states couple to the η - N channel but not to the π - N only holds to leading order in $1/N_c$, so the actual prediction is that the coupling to the π - N channel is weak for these states. In Eq. (2.3) we list two such states in the multiplet, $N_{1/2}$ and $\Delta_{3/2}$. Of these, only the $N_{1/2}$ can be clearly discerned at $N_c = 3$. Recall that at large N_c there are three $\Delta_{3/2}$ negative-parity states in the quark-shell model with a single $\ell = 1$ quark excitation, but for $N_c = 3$ there is only one. Thus, one cannot associate the physical negative-parity $\Delta_{3/2}$ state with a given K . In contrast, there are two negative-parity $N_{1/2}$ states, both at large N_c and for $N_c = 3$. Thus we predict that, to the extent the $1/N_c$ expansion is useful, one of these two states couples weakly to pions. Similarly, the other state has $K = 1$, and by an analogous argument, it should be clear that this state couples strongly to the π - N channel but weakly to the η - N channel.

How well are these predictions borne out in nature? According to the Particle Data Group [37], the $N(1535)$ has a decay fraction to the π - N channel (35–55%) that is virtually equal to its decay fraction to the η - N channel (30–55%). Now, this is striking since the phase space factor for the decay is nominally ~ 2.6 times larger for the π - N channel. Moreover, the nominal phase space obtained by assuming that the decaying particle has its Breit-Wigner mass presumably understates the relative advantage the pion has in phase space: Since the η - N threshold is only 50 MeV from the nominal mass of the $N(1535)$, if one averages over the width of the resonance in estimating the effective phase space, one substantially reduces the phase space for the η - N channel but not for the π - N channel. Thus, the $N(1535)$ clearly is much more strongly coupled to the η - N channel than to π - N . Next, consider the $N(1650)$; according to the Particle Data Group [37], $N(1650)$ has a decay fraction to the π - N channel (55–90%) that is much larger than the decay fraction to the η - N channel (3–10%). The dominance of the π - N channel is far larger than what one estimates purely from the phase space, since the π - N channel has a phase space factor ~ 1.6 times that of the η - N channel. Thus, the qualitative large N_c predictions for the dominant decay modes of the $N_{1/2}$ states are apparent in the data.

The success in describing the decay modes of the $N_{1/2}$ states tell us that the leading large N_c result does a good job in describing the mixing between these two states. One can also ask about the mixing in the context of the quark-shell model, where it is parameterized by mixing angles. Thus, another check on the phenomenological usefulness of the leading-order large N_c treatment is its ability to predict these mixing angles in a manner that is qualitatively consistent with traditional quark models. Again, we restrict our attention to those sectors for which large N_c and $N_c = 3$ have the same number of quark model states—namely the $N_{1/2}$ and $N_{3/2}$ states. From Sec. III we see that the mixing angles are given by $\theta_{N_1} = \tan^{-1}(\sqrt{2}) \approx 0.96$ and $\theta_{N_3} = \tan^{-1}(1/\sqrt{5}) \approx 2.72$ (angles in radians). In comparison, fits using decay data give $\theta_{N_1} = 0.56$, $\theta_{N_3} = 2.96$ using SU(6) [38]. The fact that N_1 mixing in large

N_c is not too far from the phenomenological one is, of course, not surprising since the latter was fit to decays; as we have seen, this is qualitatively consistent with the large N_c result for the $N_{1/2}$ states. It is encouraging, however, to note that the phenomenological fits for the $N_{3/2}$ also work well. We note also that similar results were obtained directly from $N_c = 3$ quark model calculations [39], as well as large N_c quark-shell model approaches in which $1/N_c$ corrections are included within individual operator matrix elements [25, 28, 29, 31, 32]. It will be interesting to see if similar predictions can be obtained in the strange sector [35].

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APPENDIX A: MODEL-INDEPENDENT LARGE N_c RELATIONS FOR MESON-BARYON SCATTERING

As mentioned in Sec. II, the relations between various channels in pion-baryon scattering in Eq. (2.1) were first derived in the context of a chiral soliton model [18, 19, 20, 21, 33]. However, the result is exact in the large N_c limit of QCD. In this Appendix we show how this result can be derived directly from large N_c consistency rules with essentially no model-dependent assumptions. Before doing this, we note that it has long been recognized these results are in fact model independent in the large N_c limit.

As noted long ago by Adkins and Nappi [40], the Skyrme model (treated via semi-classical projection, as appropriate for the large N_c limit of QCD) has the striking feature that many relations are completely independent of the details of the Skyrme model Lagrangian—*i.e.*, they are independent of the values of the parameters in the Lagrangian, the number and type of terms included in the Lagrangian, or even the number and types of degrees of freedom. For example, the π - N coupling is precisely $2/3$ of the π - N - Δ coupling in the leading-order treatment of *any* chiral soliton model. It was a reasonable conjecture that such relations are fully model independent and are exact results of QCD in the large N_c limit.

This conjecture was subsequently shown to be correct for all such relations tested. The method used the large N_c consistency rules discovered by Gervais and Sakita in the 1980's [4] and then rediscovered and greatly extended by Dashen and Manohar [5] in the 1990's. The only assumptions used with this method are: i) Baryonic quantities in large N_c QCD scale according to the generic large N_c rules of Witten [3], or more slowly (if there are cancellations); ii) there exists a hadronic description that reproduces the large N_c QCD results; iii) the π - N coupling scaling is generic (without cancellations), scaling as $N_c^{1/2}$; and iv) nature is realized in the most symmetric representation of the contracted $SU(4)$ group that emerges from the previous assumptions. The key to the method is to compare the scaling of π - N scattering (which scales as N_c^0) with the π - N coupling constant (which scales as $N_c^{1/2}$), suggesting that the sum of the Born term in π - N scattering plus the cross graph scales as N_c^1 . This mismatch in scattering implies the need for cancellations, and these in turn are only possible if the baryons form nearly degenerate bands of states with $I = J$ (in the two-flavor case) and lie in irreducible representations of a contracted $SU(2N_f)$. The consequences of this symmetry for various matrix elements are worked out in detail in a series of papers by Dashen, Jenkins, and Manohar [7]. One consequence is that all of the relations derived in the Skyrme model, but that are insensitive to Skyrme model details, are in fact results of large N_c QCD.

Now we note that that Eqs. (2.1) and (2.2) are, in fact, completely independent of the details of the Skyrme model, and thus one expects that it should be possible to derive it using the methods of Ref. [7]. We focus first on the case of π scattering, hence the explicit $I_\pi = 1$ below. The first step is to express the amplitudes in terms of t -channel rather than s -channel exchange. This can be done simply via a standard relation between $6j$ coefficients [41]:

$$\begin{aligned} & \left\{ \begin{matrix} K & I & J \\ R' & L' & 1 \end{matrix} \right\} \left\{ \begin{matrix} K & I & J \\ R & L & 1 \end{matrix} \right\} = \\ & \sum_{\mathcal{J}} (-1)^{I+J+L+L'+R+R'+K+\mathcal{J}} (2\mathcal{J}+1) \left\{ \begin{matrix} 1 & R' & I \\ R & 1 & \mathcal{J} \end{matrix} \right\} \left\{ \begin{matrix} R' & J & L' \\ L & \mathcal{J} & R \end{matrix} \right\} \left\{ \begin{matrix} 1 & L' & K \\ L & 1 & \mathcal{J} \end{matrix} \right\}. \end{aligned} \quad (\text{A1})$$

Inserting this identity into the first of Eqs. (2.1) yields

$$S_{LL'RR'IJ} = \sum_{\mathcal{J}} \left\{ \begin{matrix} 1 & R' & I \\ R & 1 & \mathcal{J} \end{matrix} \right\} \left\{ \begin{matrix} R' & J & L' \\ L & \mathcal{J} & R \end{matrix} \right\} s_{\mathcal{J}LL'}^t, \quad (\text{A2})$$

with

$$s_{\mathcal{J}LL'}^t \equiv \sum_K (-1)^{J+I+L+L'+K+\mathcal{J}+1} (2\mathcal{J}+1)(2K+1) \sqrt{(2R+1)(2R'+1)} \left\{ \begin{matrix} 1 & L' & K \\ L & 1 & \mathcal{J} \end{matrix} \right\} s_{KLL'}^\pi. \quad (\text{A3})$$

From the first $6j$ coefficient in Eq. (A2), it is apparent that \mathcal{J} is the isospin exchanged in the t -channel, while the second $6j$ coefficient implies that \mathcal{J} is angular momentum exchanged in the t channel. The superscript t in the function $s_{\mathcal{J}LL'}^t$ indicates that this function is given in terms of the angular momentum and isospin in the t -channel. The fact that t channel-exchanged isospin and the t channel-exchanged angular momentum are both equal to \mathcal{J} , of course, implies that they are equal to each other. Thus Eq. (A2) encodes the celebrated $I_t = J_t$ rule of Mattis and Mukerjee [20].

Now, the preceding argument shows that Eq. (2.1) implies the $I_t = J_t$ rule for pion-baryon scattering. For our purpose, the important point is that the converse is also true: The $I_t = J_t$ rule implies Eq. (2.1), since the right-hand side of Eq. (A2) is the most general form for a scattering amplitude consistent with $I_t = J_t$. Thus, if one can establish the $I_t = J_t$ rule directly from the large N_c consistency rules, then one has established Eq. (2.1) directly from large N_c QCD. However, as shown in Ref. [42] it is straightforward to establish the $I_t = J_t$ rule using the techniques of Ref. [7]. An analogous argument works for the case of η -baryon scattering.

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